technique can be employed to a noncircular large hole in a rectangular region. It can also be applied in the cases where there are more than one hole involved. Finally, eigenfunctions similar to those derived here may be utilized for the solutions of other two-dimensional problems, such as stress analyses in cylindrical shells containing large holes.

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Bending and Vibration of Transversely Isotropic Two-Layer Plates

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THE use of composite materials in various aerospace and industrial applications has prompted a considerable amount of research on the static and dynamic response of multilayer plates. During the past decade, several authors[1,2] have formulated plate theories by a direct extension of Mindlin's theory[3] for homogeneous plates. Sun and Whitney[4] have shown that laminated plate theories which are based on Kirchhoff's hypothesis, or a simple extension of Mindlin's theory, yield grossly inaccurate natural-frequency predictions for two- and three-layer plates whose layers have widely differing shear rigidities.

In a recent paper,[5] the principle of virtual work was used to derive the equations of motion in an invariant form for an arbitrary three-layered plate. No restrictions were placed on the relative thicknesses, densities, elastic moduli, or symmetries of the layers. The formulation accounts for the shear deformation of each layer as well as the translational and rotational inertia of the composite. Continuity of displacements and stresses was imposed in accordance with a perfect interface bond assumption.

In the current analysis, the previously derived equations will be used to analyze a transversely isotropic two-layer plate by deleting the terms associated with the third layer and neglecting the transverse contraction of the composite. The theory then becomes the two-dimensional analog of Theory II as presented by Sun and Whitney.[4] If we assume that each layer is transversely isotropic, the equations of motion are written in vector notation and can be uncoupled to yield a sixth-order equation in the transverse displacement. By neglecting certain in-plane and rotatory inertia terms, we can obtain a somewhat simpler fourth-order equation, which is very similar to Mindlin's dynamic plate equation with modified stiffness, mass, and inertia coefficients. This equation reduces to Mindlin's formulation if either of the layers is assumed to vanish or the properties of both layers are identical. From virtual work, the natural boundary conditions can also be derived in an invariant form for a plate of arbitrary shape.[5]

To substantiate the differences between the current work and the formulations that are in present use, natural-frequency calculations have been performed for a variety of layer property configurations. The numerical results obtained for transversely isotropic layers indicate that laminated theory[7] based on Kirchhoff's hypothesis will, in general, be in substantial error, even for relatively thin plates, if the ratios of the in-plane-to-transverse shear moduli of the two layers are large.

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Governing Equations

The equations of motion are developed for an elastic, transversely isotropic two-layer plate in the state of generalized plane stress. It is assumed that the in-plane displacements vary linearly through the thickness of each layer, however, the cross-sectional rotations of the layers are not necessarily equal. The transverse displacement is assumed to be constant through the plate thickness and the transverse normal stress is assumed to be zero.

The equations of motion can easily be deduced from previously derived equations.[5] When both layers are transversely isotropic then, as before, vector notation can be employed to describe all the equations. It is also possible to uncouple the field equations, which leads to a sixth-order partial differential equation in the transverse displacement w. It can be shown that neglecting certain in-plane and rotatory inertia terms results in the following simplified uncoupled equation

$$B_1\nabla^4 w + B_2\nabla^2 w,_{tt} + B_3 w,_{tttt} - \bar{m}\sum_{n=1}^{2}\overset{(n)}{b}\, w,_{tt} + \sum_{n=1}^{2}\overset{(n)}{b}\, p_3 + B_4\nabla^2 p_3 - (B_3/\bar{m})p_{3,tt} = 0 \qquad (1)$$

where

$$B_1 = -\Big[\sum_{n=1}^{2}(\overset{(n)}{b}\,\overset{(n)}{h})^2 + \overset{(1)}{b}\,\overset{(2)}{b}\Big(4\sum_{n=1}^{2}\overset{(n)}{h}{}^2 + 6\,\overset{(1)}{h}\,\overset{(2)}{h}\Big)\Big]/12$$

$$B_2 = \Big\{\sum_{n=1}^{2}\overset{(n)}{\rho}\,\overset{(n)}{b}\,\overset{(n)}{h}{}^3 + \overset{(2)}{\rho}\,\overset{(1)}{b}\,\overset{(2)}{h}\,(6\,\overset{(1)}{h}{}^2 + 9\,\overset{(1)}{h}\,\overset{(2)}{h} + 4\,\overset{(2)}{h}{}^2)]$$
$$- \overset{(1)}{\rho}\,\overset{(2)}{b}\,\overset{(1)}{h}{}^2(2\,\overset{(1)}{h} + 3\overset{(2)}{h})$$
$$+ (\bar{m}/h_T)\Big[\sum_{n=1}^{2}(\overset{(n)}{b}{}^2\,\overset{(n)}{h}{}^2/\overset{(n)}{a}) + \overset{(1)}{b}\,\overset{(2)}{b}\Big]\ \Big\{(\overset{(1)}{a}\,\overset{(2)}{h}{}^2$$
$$+ \overset{(2)}{a}\,\overset{(1)}{h}{}^2)]/\overset{(1)}{a}\,\overset{(2)}{a} + 3\,\overset{(1)}{h}\,\overset{(2)}{h}\,(\overset{(1)}{a} + \overset{(2)}{a})$$
$$/\,\overset{(1)}{a}\,\overset{(2)}{a}\ \Big\}\Big]\Big\}/12$$

$$B_3 = (\bar{m}/12h_T)\Big\{-\sum_{n=1}^{2}\overset{(n)}{\rho}\,\overset{(n)}{b}\,\overset{(n)}{h}{}^3/\overset{(n)}{a} - (\overset{(2)}{\rho}\,\overset{(1)}{b}\,\overset{(2)}{h}/\overset{(2)}{a})$$
$$(6\,\overset{(1)}{h}{}^2 + 9\,\overset{(1)}{h}\,\overset{(2)}{h} + 4\,\overset{(2)}{h}{}^2)] + (\overset{(1)}{\rho}\,\overset{(2)}{b}\,\overset{(1)}{h}{}^2$$
$$/\,\overset{(1)}{a})\,(2\,\overset{(1)}{h} + 3\,\overset{(2)}{h})\Big\}$$

$$B_4 = -\Big\{\sum_{n=1}^{2}\overset{(n)}{b}{}^2\,\overset{(n)}{h}{}^2/\overset{(n)}{a}$$
$$+ \overset{(1)}{b}\,\overset{(2)}{b}\Big[4\sum_{n=1}^{2}\overset{(n)}{h}{}^2/\overset{(n)}{a}$$
$$+ 3\,\overset{(1)}{h}\,\overset{(2)}{h}\sum_{n=1}^{2}\overset{(n)}{a}/\overset{(1)}{a}\,\overset{(2)}{a}\Big]\Big\}/12h_T$$

Here $\overset{(n)}{h}$, $\overset{(n)}{\rho}$ denote the thickness and density of layer "n," respectively;

$$h_T = \overset{(1)}{h} + \overset{(2)}{h},\ \bar{m} = \sum_{n=1}^{2} \overset{(n)}{\rho}\ \overset{(n)}{h},$$

and p_3 is the transverse surface loading.

Also,

$$\overset{(n)}{a} = \overset{(n)}{G}_{32} \text{ and}$$

$$\overset{(n)}{b} = \overset{(n)}{h}\ \overset{(n)}{E}_{22}/(1-\overset{(n)}{\nu}{}_{12}^{2}),$$

where $\overset{(n)}{G}_{32}$, $\overset{(n)}{E}_{22}$, $\overset{(n)}{\nu}_{12}$ are the transverse shear modulus, in-plane Young's modulus, and in-plane Poisson's ratio of layer "n," respectively.

Free and Forced Harmonic Vibrations

In this case, Eq. (1) becomes

$$B_1\nabla^4 W - B_2\omega^2\nabla^2 W + \omega^2(\bar{m}\sum_{n=1}^{2}\overset{(n)}{b} + \omega^2 B_3)W$$
$$= -[\sum_{n=1}^{2}\overset{(n)}{b} + \omega^2(B_3/\bar{m})]Q - B_4\nabla^2 Q \qquad (2)$$

where W and Q are amplitudes of w and p_3, respectively, and ω is the frequency of harmonic vibration. For a simply supported rectangular plate, the boundary conditions are $w=0$, $\nabla^2 w=0$ along the contour. This yields, in conjunction with Eq. (2) and for $Q\equiv 0$, the following equation for the frequency of flexural vibrations:

$$\omega_{mn}^2 = -2B_1 D_{mn}^2/\{\bar{m}\sum_{n=1}^{2}\overset{(n)}{b} + B_2 D_{mn}$$
$$+[(\bar{m}\sum_{n=1}^{2}\overset{(n)}{b} + B_2 D_{mn})^2 - 4B_3 B_1 D_{mn}^2]^{1/2}\} \quad (m,\ n=1,2...) \qquad (3)$$

where $D_{mn}=(m\pi/L_x)^2+(n\pi/L_y)^2$ and L_x, L_y are the plate dimensions. Although there are two roots of the frequency equation, the other root is associated with a thickness-shear type of mode and is not correct since several in-plane and rotatory inertia terms have been discarded in the process of uncoupling the equations of motion.

Introducing the dimensionless frequency

$$\Omega_{mn} = \omega_{mn}^2 L_y^4\ \bar{m}/\pi^4\ \overset{(1)}{b}\ \overset{(1)}{h}{}^2 \qquad (4)$$

and expressing Ω_{mn} in terms of the basic dimensionless parameters described below, one obtains

$$\Omega_{mn} = -2K_1^2 Y_1/[1+K+Y_2+[(1+K+Y_2)^2 - Y_1 Y_3]^{1/2}] \qquad (5)$$

where

$$\overset{(n)}{A} = \overset{(n)}{E}_{22}/\overset{(n)}{G}_{32}(1-\overset{(n)}{\nu}{}_{12}^{2}),\ F=\overset{(2)}{G}_{32}/\overset{(1)}{G}_{32},\ K=\overset{(2)}{b}/\overset{(1)}{b},\ N=L_y/L_x,$$

$$P=\overset{(2)}{h}/\overset{(1)}{h},\quad R=L_y/\overset{(1)}{h},\quad S=\overset{(2)}{\rho}/\overset{(1)}{\rho},\quad K_1=m^2N^2+n^2$$

$$K_1 = m^2N^2+n^2$$
$$Y_1 = -[1+4K+KP(6+4P+KP)]/12$$
$$Y_2 = \pi^2 K_1\{[(1+SP(KP^2+4P^2+9P+6)-3P-2)/(1+SP)] + [\overset{(1)}{A} + \overset{(2)}{A} P(KP+4(F+P)+3P(1+F))]/(1+P)\}/12R^2$$
$$Y_3 = \pi^4 K_1\{[\overset{(1)}{A}(F+PS(4P^2+9P+6)) + \overset{(2)}{A} F(SP^3+2F)-3PF^2]/[3R^4F(1+P)(1+SP)]\}$$

Numerical Examples and Comparisons

Recent work[6] has indicated that transverse isotropy has a major effect on the static and dynamic response of homogeneous beams and plates. When the in-plane-to-transverse shear modulus ratio is large ($E/G>20$), the deflection and natural-frequency predictions of classical isotropic and orthotropic homogeneous plate theories are found to be in substantial error. Since many recently developed transversely isotropic materials have an E/G value of 200 or larger,[6] one might anticipate that classical laminated plate theory would be inadequate in dealing with such materials.

In an effort to illustrate the differences between the current analysis and classical laminated plate theory,[7] the problem of

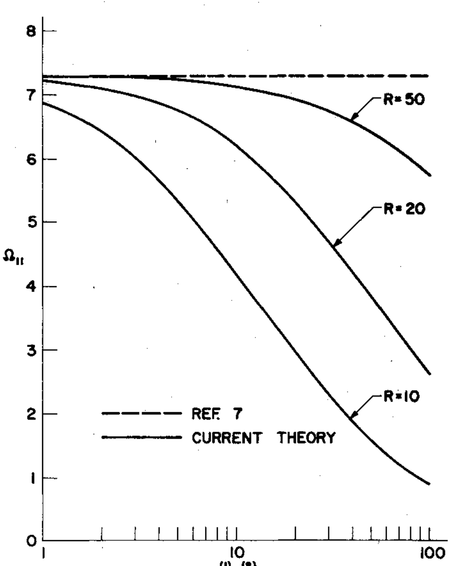

Fig. 1 Dimensionless fundamental frequency ($F=K=10$, $N=P=S=1$).

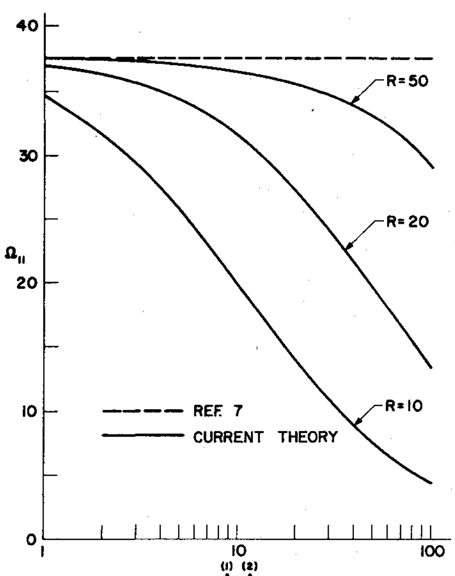

Fig. 2 Dimensionless fundamental frequency ($F=K=100$, $N=P=S=1$).

free vibrations of a simply supported square plate consisting of two transversely isotropic layers was studied for several in-plane Young's moduli, in-plane-to-transverse shear moduli, and length-to-thickness ratios. The natural frequency results for the fundamental mode ($m = n = 1$) are shown in Figs. 1 and 2.

As demonstrated by the figures, the in-plane-to-transverse shear moduli ratio has a very pronounced effect on the flexural frequencies, especially for thick composite plates. Even for plate length-to-thickness ratios of 20 or more, significant differences exist between the two predictions. It should be noted that, as the ratio of the in-plane Young's moduli increases, and therefore the ratio of the shear moduli increases since ($\overset{(1)}{A} = \overset{(2)}{A}$), the discrepancies become somewhat larger. Although not shown here for the sake of brevity, the discrepancies become much larger when higher modes are considered.

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Inviscid Flow Past a Sharp-Nosed Body with a Closed Finite Wake

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THE most commonly used model for the description of infinite Reynolds number flows past bodies whose shapes cause flow separation and the formation of a wake is the Helmholtz[1]-Kirchhoff[2] free streamline model. Batchelor[3] raises objections to both the open and the closed wake forms of this theory and proposes instead a closed wake model with a recirculating flow inside the wake which is such as to satisfy the requirement of continuity of pressure across the wake boundary. However, because the pressure is not constant along the boundary, much of the classical potential flow theory for wakes and cavities is not applicable. It is, consequently, much more difficult to determine the shape of the wake bubble than it is with the free streamline theory.

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In the present work a method is presented for computing, using Batchelor's model, the infinite Reynolds number flow for the case where the body is a two-dimensional shell forming the front part of a slender wake bubble.

Mainstream Irrotational Flow

Closed wake bubbles behind two-dimensional objects must necessarily terminate with a cusp to allow the velocity just outside the bubble boundary to be uniform. For simplicity, the present investigation is restricted to two-dimensional flows with fore and aft symmetry. In this case the wake bubble must have a cusp at its upstream end as well as its downstream end. The flow in the interior of the bubble is characterized by uniform vorticity of strength $-\omega$ in the upper half of the bubble and $+\omega$ in the lower half.[3,4] Thus, with distances and velocities made dimensionless with respect to the bubble length and the uniform stream velocity, respectively, the flow configuration is as shown in Fig. 1 (the ζ-plane), where t is the thickness ratio for the bubble, (x, y) are Cartesian coordinates and u_x and u_y are the velocity components in the x and y directions, respectively.

A major simplifying assumption is now made by supposing the bubble to be a slender one. That is, t is assumed small. With this simplification the magnitude of the velocity along the outside of the bubble boundary (ABC in Fig. 1) will be little different from the uniform stream value of unity. The flow in the ζ-plane may be determined through a mapping of the flow past a unit-diameter circular cylinder in the z-plane. The problem of finding the appropriate conformal transformation is, in general, a difficult one. However, since t is small, the required mapping may be regarded as a small perturbation of the mapping of the cylinder flow in the z-plane to the flow past a flat plate (AOC in Fig. 1) in the ζ-plane. A suitable form for the transformation is therefore

$$\zeta = z + \frac{1}{4z} - t\sum_{j=1}^{N} A_j (2z)^{1-2j} \tag{1}$$

where

$$\zeta = x + iy \qquad z = re^{i\theta}$$

and where the $A_1 - A_N$ are real coefficients to be determined later by matching the irrotational flow to the inviscid flow in the wake.

The cylinder surface $r = ½$ maps onto the bubble surface. Hence, equating real and imaginary parts of the transformation [Eq. (1)], putting $r = ½$, and eliminating θ give the following equation for the shape of the bubble surface

$$y = t(1 - x^2)^{½} \sum_{j=1}^{N} A_j U_{2j-2}(x) + 0(t^2) \tag{2}$$

where $U_{2j-2}(x)$ is the Chebyshev polynomial of the second kind in x of degree $2j-2$. Since, when $x = 0$, $U_{2j-2} = (-1)^{j-1}$, and $y = t$, there is the requirement that

$$1 = \sum_{j=1}^{N} (-1)^{j-1} A_j \tag{3}$$

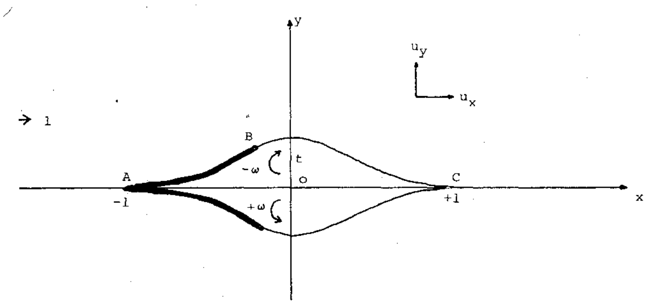

Fig. 1 Flow configuration (the ζ-plane).